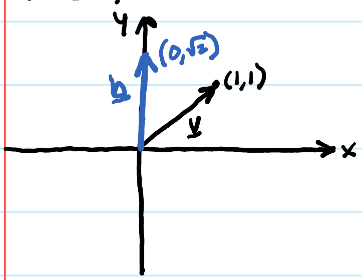
Eigenvector Review-1

Review of Eigenvalues and Eigenvectors:

* Typically, a matrix is thought of as
$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{N1} & \cdots & \cdots & A_{NN} \end{bmatrix}$$

* But, matrices can also be thought of based on what they do. For example:



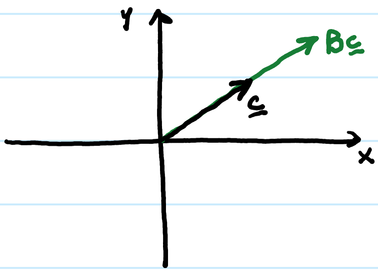
$$\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A\underline{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} = \underline{b}$$

* Matrices act on vectors to produce another vector. A can also be called a "linear opperator."

<u>Eigenvectors:</u> What is there was a vector $\underline{c}$ that when acted on by a matrix B produced a vector pointing in the same direction as $\underline{c}$?

$B\underline{c} = \lambda\underline{c}$, λ is some scalar



* Rewrite $\lambda\underline{c}$ as $\lambda I\underline{c}$ where I is the identity matrix.

$$B\underline{c} = \lambda I\underline{c}$$

$$(B - \lambda I)\underline{c} = \underline{0}$$

Trivial solution: $\underline{c} = \underline{0}$

* For non-trivial $\underline{c} \neq \underline{0}$, what should be true about $B - \lambda I$?

Eigenvector Review-2

__Aside:__ consider the matrix $D = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ Where do the vectors $D\underline{l} = \underline{m}$ lie?

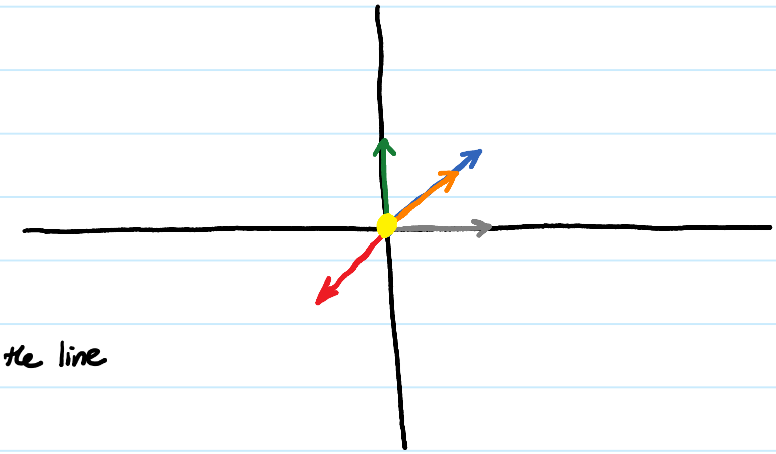
$\underline{l} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\underline{m} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{l} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\underline{m} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\underline{l} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\underline{m} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\underline{l} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\underline{m} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{l} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\underline{m} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



* So, D maps 2-D onto the line $y = x$

* If a vector is parallel to the line $y = x$, D sends that vector to $\underline{0}$. This is what we want!!

* What is special about D?
  - The columns of D are scalar multiples of each other
  - The determinant of D $\det(D) = 0$

Now, back to $(B - \lambda I)\underline{c} = \underline{0}$. For nontrivial $\underline{c}$, $\det(B - \lambda I) = 0$

__Ex__ $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

Find eigenvalues and eigenvectors of B

$\det(B - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 2-\lambda & -1 \\ 0 & 1-\lambda \end{bmatrix}\right) = 0 \Rightarrow (2-\lambda)(1-\lambda) = 0$

$\lambda = 1, 2$

__$\lambda = 1$__

$\begin{bmatrix} 2-1 & -1 \\ 0 & 1-1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 - c_2 = 0 \Rightarrow c_1 = c_2$

$\boxed{\lambda = 1, \ \underline{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

Eigenvector Review-3

$\underline{\lambda=2:}$

$$\begin{bmatrix} 2-2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -c_2=0 \\ c_2=0 \end{matrix}$$

$$\boxed{\lambda=2,\ c=\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$\underline{\text{Ex}}$ $B=\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}\right)=0 \Rightarrow (3-\lambda)(2-\lambda)-2=0$$

$$6-5\lambda+\lambda^2-2=0$$

$$\lambda^2-5\lambda+4=0$$

$$(\lambda-4)(\lambda-1)=0$$

$$\lambda=1,4$$

$\underline{\lambda=1:}$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2c_1+c_2=0 \qquad c_2=-2c_1$$

$$\boxed{\lambda=1,\ \underline{c}=\begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$\underline{\lambda=4:}$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -c_1+c_2=0 \\ 2c_1-2c_2=0 \end{matrix} \qquad c_1=c_2$$

$$\boxed{\lambda=4,\ c=\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Eigenvector Review-4

<u>Eigenvalue decomposition:</u>

$$B\underline{c} = \lambda \underline{c}$$

Define $C = [c_1 | c_2 | \ldots | c_n]$ and $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$

and rewrite as

$$BC = C\Lambda$$

Post multiply both sides by the inverse of C, C^{-1}

$$BCC^{-1} = C\Lambda C^{-1}$$

$$\boxed{B = C\Lambda C^{-1}}$$

Ex) $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$C^{-1} = \frac{1}{0-1}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow C^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \checkmark$$

Eigenvalue decomposition is incredibly useful for evaluating things like B^n

Ex) $B^3 = (C\Lambda C^{-1})^3 = (C\Lambda C^{-1})(C\Lambda C^{-1})(C\Lambda C^{-1})$

$= (C\Lambda C^{-1})(C\Lambda C^{-1}C\Lambda C^{-1})$

$= (C\Lambda C^{-1})(C\Lambda I \Lambda C^{-1})$

$= (C\Lambda C^{-1})(C\Lambda^2 C^{-1})$

$= C\Lambda C^{-1}C\Lambda^2 C^{-1} \Rightarrow C\Lambda^3 C^{-1} \Rightarrow$ $\boxed{B^3 = C\Lambda^3 C^{-1} \quad \Lambda^3 = \begin{bmatrix} \lambda_1^3 & & 0 \\ & \lambda_2^3 & \\ 0 & & \ddots \; \lambda_n^3 \end{bmatrix}}$